Abstract

Dynamic representations help make accessible Omar Khayyam’s marvelous, yet seldom-seen, approach to solving one form of a cubic equation, thus supporting students’ engagement with of Khayyam’s 11th-century solution in an undergraduate course in the history of mathematics.

Introduction

A common goal for a general survey course in the history of mathematics is for students to learn something about the historical origins of mathematics. One approach to this is to visit “the first” known occurrence of a collection of now well-known results and the mathematical practitioners who realized them. One risk of presenting the development of mathematics as a highlight reel is that students may miss the fact that mathematics develops in particular social and historical contexts perhaps featuring mathematical understanding or practices very different from our own. Another unfortunate possibility of an exclusively “greatest hits” approach to the history of mathematics is the communication of an inaccurate message that mathematics grows monotonically: groundbreaking results somehow inevitably stack up, one neatly atop another, to arrive at our modern mathematical understanding. Both of these tendencies can be difficult to combat, particularly in a one-semester course tasked with covering mathematics from Plato through NATO — or at least up to the nineteenth century.

Studying primary sources from the history of mathematics is one way to foster an understanding of mathematics as an ongoing, creative process that occurs in specific times and places. Grappling with the contours of these contexts can be crucial in really understanding innovation or novelty and for increasing appreciation of the insights and imagination that contribute to the development of mathematical results and theories. Primary sources can also illuminate gradual change, including incremental shifts in perspective that do not persist as part of our current mathematical landscape. Exposure to what may be seen as mathematical missteps or false starts can also enrich student understanding of mathematical activity. Such historical investigations additionally expand student understanding of historiographic issues and of the kinds of questions historians ask and the tools they use to answer them.

Although primary source material provides these rich opportunities for mathematical and historical insights, students often find it frustrating to deal with archaic sources that originate in a profoundly different worldview. Linguistic and notational differences pose challenges, too. Students are sometimes uninterested in overcoming these difficulties to explore an historical approach to a particular mathematical problem, especially when they are acquainted with simpler, more efficient, or more general modern methods. This can be particularly evident in instances of studying solutions for polynomials of small degree, where the realities of studying primary source translations can obscure the remarkable work of historical mathematicians.

In this paper, we provide an example of how technology can benefit student engagement and facilitate greater appreciation for both the content and context of historical work. The example explored below is based directly on translated source material of Omar Khayyam’s eleventh-century approach to constructing a solution to a cubic. The GeoGebra applet included here aims to make accessible Khayyam’s marvelous, yet seldom-seen, approach. This dynamic representation eliminates the difficulty of visualizing and interpreting a static image of Khayyam’s proof, thus surmounting some of the obstacles to and maintaining the benefits of including
Khayyam’s solution in an undergraduate course in the history of mathematics.

**Polynomials in History of Mathematics Courses**

Investigating the treatment of what we call polynomial functions can provide a nice coherence to material from ancient Greece through Niels Henrik Abel’s nineteenth-century proof that no general solution exists for a quintic. Students may first encounter *igi-igibum* problems from ancient Mesopotamia and then move forward in time to grapple with Greek geometry, including methods for extracting square roots and completing the square.

![Figure 1: An *igi igibum* problem, or “completing the square,” illustrated by Eleanor Robson (Robson, p. 183).](image)

These topics give students access to thinking about how mathematical practitioners throughout history have dealt with similar kinds of problems. These investigations also can be used to highlight particular features of the historical context, perhaps a view of the purpose of mathematics, or the role of abstraction, or the nature of mathematical justification. The investigation of equation-solving also raises significant questions about the nature of notation and the development of algebra. Since algebra is a foundational component of modern mathematics very familiar to undergraduates, it can provide a helpful touchstone throughout a history of mathematics course. Along the way, students encounter the work of Diophantus, al-Khwarizmi, Cardano, Viete, Descartes, Recorde, Newton, Euler, Lagrange, Gauss, Ruffini and many others as a notion emerges of algebra as a generalized problem-solving tool.

While the modern framework of algebraic solutions to polynomials can aid student understanding, it can also hinder deeper appreciation for the nuance of historical mathematics in
context. So familiar is the modern notion of a quadratic of the form $ax^2 + bx + c = 0$ solvable by the quadratic formula, that it is easy to read modern ideas into an historical text. It is indeed a challenging mental exercise to think in a context that allows only positive real coefficients —
a place where $x^2 + bx = c$ fundamentally differs from $x^2 + c = bx$ — or to imagine a mathematical world where $x^2$ literally represents a square and dimensionality fundamentally matters in a polynomial-like expression. Yet, appreciating historical mathematical work sometimes poses these challenges. Teaching an appreciation of historical mathematical work poses additional challenges.

Ideally, an instructor would like to minimize difficulties so that students may engage in productive effort. One approach to achieving this is to make careful connections between historical mathematics and mathematics students are more familiar with. For example, students generally understand that solutions to equations of the form $ax^3 + bx^2 + cx + d = 0$ can be generated from the coefficients of the cubic. Even for students unfamiliar with the intricacies of deriving such a formula, this may seem a natural extension of the quadratic formula and, more broadly, the notion of having an algorithm to deal with a certain kind of problem is a familiar one. What may feel very unnatural is the way historical practitioners approached these problems. Below, we focus on a constructive solution to the cubic as one example from the richly textured history of work on finding solutions to cubic equations. One feature of this dynamic representation is the illustration of a relationship between an 11th-century result and a modern Cartesian graph.

**A Common Cubic Narrative**

In a general History of Mathematics survey course, the work of Girolomo Cardano is a common focal point for studying a general solution to the cubic. Cardano’s well-known solution to the cubic appeared in *Ars Magna* in 1545 and is often credited as the first publication of a general solution to cubic equations.

![Cardano's Ars Magna Title Page](http://www.maa.org/publications/periodicals/convergence/mathematical-treasures-emars-magnaem-title-page#sthash.z4fJ7BRM.dpuf)

Figure 4: The title page of Cardano’s *Ars Magna* from http://www.maa.org/publications/periodicals/convergence/mathematical-treasures-emars-magnaem-title-page#sthash.z4fJ7BRM.dpuf.

Students usually encounter Cardano’s solution either with 3D manipulatives of the type described by William Branson, or in terms of an algebraic formulation as done by William Dunham...
Not only does Cardano’s method offer a very satisfying intersection of algebraic and geometric methods, but the related historical narrative is rich with drama and intrigue, briefly outlined below.

The first algebraic solution to a cubic equation is due to Scipione del Farro, a fifteenth-century mathematics professor at the University of Bologna. At some point, del Farro shared his guarded knowledge — a method of solving problems of the form “cube and things equal to numbers,” or \( x^3 + bx = c \) with \( a, b > 0 \) — with his student Antonio Maria Fióre, who proceeded to make a living by challenging others to mathematical problem-solving contests. Since Fióre knew del Farro’s secret, he always posed problems of the form \( x^3 + bx = c \). Fióre’s trick worked until he challenged Niccoló Tartaglia in 1535.

![Figure 5: Niccoló Tartaglia, (1499-1557), from Convergence Portrait Gallery.](image)

Tartaglia developed his own solution and won the contest (although he declined Fióre’s prize of 30 free dinners). Word of Tartaglia’s fame and accomplishment reached Giorolomo Cardano in Milan. After a long campaign, Cardano convinced Tartaglia to meet with him in 1539 to divulge the secret in verse on the condition that Cardano swear not to publish it. Cardano and his student Ludovico Ferrari subsequently learned that del Farro knew the solution method. Since they also developed the material further, discovering how to solve other types of cubics, and, thanks to Ferrari, also quartics, they decided to publish this work in *Ars Magna* in 1545. The enraged Tartaglia fought bitterly against Cardano and Ferrari for years afterwards.

The hint Cardano initially got from Tartaglia worked only for a special class of cubics of the form \( x^3 + ax = b \) with \( a, b > 0 \). For contemporary readers, of course, \( ax^3 + bx^2 + cx + d = 0 \)
encompasses all possible cases, where the coefficients are positive, or negative, real, or imaginary. Prior to the widespread adoption of negative numbers, though, mathematical practitioners considered each possible combination of coefficients for a cubic equation. It is, naturally, difficult (if not impossible) for students to imagine practicing mathematics without the generalized problem-solving tools of modern algebra. It is likewise often challenging for students to appreciate how, in context, Tartaglia had actually given Cardano the solution to one type of cubic. Cardano then continued working with Ferrari to extend the result to include other types of cubics. In total, they investigated fourteen types of cubic equations. Due to these efforts, Cardano’s name is now attached to the three equations that give the three roots of the modern cubic equation $ax^3 + bx^2 + cx + d = 0$, which encompasses all the various forms of medieval cubics.

Although Cardano gets significant credit, he and his fellow Italian mathematicians were certainly not the first to grapple with solutions of cubic equations. Some ancient Babylonian tablets from the 20th-16th centuries BCE have tables for calculating cubes and cube roots, although there is no evidence they used these tables to solve equations. The doubling of a cube is a well-known problem of classical antiquity attempted by mathematical practitioners including Hippocrates, Menaechmus, and Archimedes, although it is extremely unlikely that any of them formulated the problem in terms of a cubic equation (Aminrazavi and Van Brummeln). The *Nine Chapters on the Mathematical Art*, a classic Chinese text compiled around the 2nd century BCE includes some methods for solving some cubic equations. In 7th century China, Wang Xiaotong solved 25 cubic equations of the form $x^3 + px^2 + qx = N$ (Mikami, pp. 53-56). In the 11th century, Omar Khayyam contributed a geometric solution to the theory of cubic equations. In
twelfth-century India, Bhaskara II attempted the solution of cubic equations. In Persia, Sharaf al-Din al-Tusi treated several types of cubic equations in his Treatise on Equations. Yet another twelfth-century investigation came from Leonardo of Pisa (Fibonacci), who found the positive solution to a particular cubic equation using Babylonian numerals. The treatment of cubics additionally occupied many other mathematical practitioners from the 9th through the 16th centuries (Aminrazavi and Van Brummelen).

These considerable efforts are often overlooked in undergraduate history of mathematics survey classes. There are many defensible explanations for this, ranging from lack of space in a semester syllabus to the challenges — linguistic, cultural, or mathematical — posed by the material itself. The GeoGebra applet included here aims to highlight one of these seldom-seen investigations in an accessible way. Omar Khayyam’s constructive solution to the cubic serves as one example from the richly textured history of work on finding solutions to cubic equations. The use of GeoGebra software removes the difficulty of interpreting Khayyam’s static diagram, enabling students to focus on appreciating the construction and understanding the justification that the constructed segment is, in fact, a solution. This dynamic visualization also links Khayyam’s construction with a modern Cartesian graph of a cubic equation. Students can interact with this dynamic representation, making Khayyam’s remarkable work more accessible and engaging, and thus easier to integrate in a History of Mathematics course.

Omar Khayyam and cubic equations

Khayyam’s full name, Abū’l-Fath Ghiyāth al-Dīn 'Umar ibn Ibrāhīm al-Khayyāmī al-Nishāpūrī, suggests that his family trade was making tents, but he is perhaps best remembered as a poet, and also as a mathematician and philosopher from a region near present-day Afghanistan.

Figure 7: Omar Khayyam, (1044-1123), from Convergence Portrait Gallery.

He may be most well-known for his *Rubā'iyāt* quatrains which critique the fundamental tenets of religion through discussion of issues such as epistemology, eschatology, determinism, and quest
for meaning. In mathematics, Khayyam worked within an Islamic tradition of investigating Euclid’s parallel postulate and discussing the definition of ratios. In addition to producing a thorough investigation of cubics, Khayyam additionally discovered a general method for root extraction of arbitrarily high degree. He appears to have arrived at questions about cubic equations through his investigation of geometrical problems.

In a translated excerpt of Khayyam’s work, he discussed a “square square,” or, for modern readers, an $x^4$ term, as something that “does not exist in reality in any way,” but instead in the realm of philosophy (Fauvel and Gray, p. 226). He continued to say “whatever is obtained by algebra is obtained” by four things: “number, object [$x$], square [$x^2$], and cube [$x^3$].” Dimensionality provided the justification for this. Khayyam specified that number is “a state of mind independent of all magnitudes,” while a straight line denoted the object familiar to modern readers as $x$. A square, which we denote, $x^2$, Khyyam “denoted by a quadrilateral of equal sides with right angles whose side is equal to a straight line called object,” which we would call $x$. And, finally, he described a cube, known to us as $x^3$, as “a solid which is bounded by six equal surfaces of four sides whose sides are equal, angles are right angles.” Those four sides were each straight line objects, or, for us, $x$ and each surface was a square, in our notation, $x^2$. Khayyam referenced Euclid’s *Elements* (XI, 27) for the construction of a cube and asserted that “an object with more than three dimensions is impossible” (Fauvel and Gray, p. 226).

Khayyam stated that methods for finding unknowns for the six different forms of quadratic equations have been thoroughly “explained in books of algebraists” (Fauvel and Gray, p. 226). In fact, ruler and compass constructions to solve quadratic equations and similar methods date back as far as the Greeks. Khayyam would have been familiar with both the *Elements* of Euclid and the *Conics* of Apollonius. Cubic equations, however, do not appear in these sources. Khayyam enumerated the following types of cubic polynomials. He used the language of “a cube equal to squares” for what the modern reader will recognize as $x^3 = ax^2$, or “a cube plus edges equal to squares and numbers” for $x^3 + ax = bx^2 + c$ and so on, but we adopt polynomial notation here.
Because the coefficients $a$, $b$, and $c$ must be greater than zero for Khayyam, these represented different forms of cubic equations. Equations 1, 2, 4, 8, and 9 can be reduced to quadratic equations and thus were solvable by methods from Euclid’s *Elements*, Book II. Finding solutions for the remaining fourteen cubic equations required solid geometry, specifically conics and conic sections.

To contemporary students, a geometric solution to a cubic equation may seem strange. Exploring this approach poses a challenge of communicating how geometric problems motivated the study of cubic equations. Medieval Islamic algebra is, in many ways, a foreign terrain both conceptually and notationally. We do not have access to sources that reveal the particular insight that led to Khayyam’s construction. In fact, this is a common situation when dealing with historical mathematics. Rarely do we have access to the developmental process of historical practitioners of mathematics. One exceptional example is the recent discovery of a previously unknown text *The Method* by Archimedes that resolved centuries of questions about how he generated the formulas he proved by methods of contradiction. For now, we have no such insight in the case of Khayyam. This foray into medieval Islamic mathematics nonetheless invites discussion of the mathematical environment and tools available to Khayyam — as well as how they differed from those of Cardano and others. Encountering the perspective that powers of $x$ correspond to actual geometrical dimensions is also a valuable addition to a students’ experience of historical mathematics. In this context, bounded in ways by three dimensions of physical space, the solution of a cubic equation marked a significant achievement.

**Khayyam’s construction**

Khayyam began by providing directions from which the reader can construct a line segment of length that gives the solution to a polynomial of the form “cubes and squares and roots equal
a number” (Fauvel and Gray, p. 233). Note that “cubes” corresponds to a modern $x^3$ term, and “squares” would be, for us, $x^2$, and “roots,” then, for readers, $x$, so modern notation expresses this sum of cubes, squares, and roots equal to a number in the form $x^3 + C_1 x^2 + C_2 x = C_3$. Since Khayyam’s mathematical context viewed only objects of like dimension as additive, each term in this equation would need to involve three powers. A contemporary articulation of this would be of the form $x^3 + ax^2 + b^2 x = c^3$. (For example, in the cubic $x^3 + 4x^2 + 6x = 8$, values for the construction would be $a = 4, b = \sqrt{6}, c = 2$.) The following set of instructions constructs a line segment of length $x$, a solution to a cubic equation of the form $x^3 + ax^2 + b^2 x = c^3$.

![Figure 8: Khayyam’s construction, in which line segment LB is the solution to the cubic.](image)

First, Khayyam says to construct a line segment $HB$ of length $b$, the square root of the given number of edges. Then build a rectangular solid of volume $c^3$ whose base is a square with sides of length $HB$. Take the height of this solid to be $BG$ and construct a segment $BG$ perpendicular to $HB$. Since the base is $b^2$ and the volume $c^3$, then in terms of coefficients of the modern polynomial, $BG$ is of length $c^3/b^2$. The final piece of the construction utilizing given information is to put point $D$ in line with $GB$ such that $DB$ is a segment of length $a$.

Next, construct a semi-circle with diameter $GD$. In Khayyam’s diagram, this is the semi-circle in the upper half plane. The segments $GB$ and $HB$ then determine a rectangle that is completed by adding point $K$ and segments $GK$ and $HK$. Draw then a hyperbola passing through point $G$ with asymptotes $HK$ and $HB$. This hyperbola will intersect the semicircle at a known point $Z$. Drop a line from $Z$ perpendicular to $GD$ that intersects $GD$ at point $L$ and intersects $KH$ at point $T$. The line segment $LB$ is the solution to the cubic. At this point, although Khayyam has constructed the segment which satisfies the given cubic, his instructions continue to complete the geometrical constructions necessary to justify $LB$ as a solution. So draw a line through $Z$ at a right angle to the $ZLT$ line just constructed. Extend line $HB$ to intersect that line at point $A$. These rectangles conclude Khayyam’s construction.
Intermingled with the above instructions is a justification that the constructed line segment indeed satisfies the cubic. A presentation of his argument is below. Key steps are highlighted and explanations of geometric facts perhaps less familiar to modern readers are included. The main tools in Khayyam’s argument are ratios and geometric reasoning, and the properties of the hyperbola and the properties of the circle.

Khayyam’s Geometric Demonstration

The construction gives instructions from which the reader can construct a segment of length that gives the solution to a polynomial of the form “cubes and squares and roots equal a number.” Because mathematical practice of the time insisted on maintaining dimensionality, it would only be possible to add terms with like dimension, so each term in the polynomial would essentially be viewed as a three-dimensional rectangular box. For Khayyam, then, finding the solution of a cubic was a geometrical question of finding the right sized line segment \((LB)\) with which three boxes can be built so that their combined volumes equal a given volume. The first box is a perfect cube, \((LB)^3\). The second box has a square base with area \((LB)^2\) and given height \(DB\). The third box has height \(LB\) and given square base \((HB)^2\). The total volume of these three boxes must be equal to the given volume \((HB)^2 \times BG\). This condition generates a cubic in \(LB\) and Khayyam’s construction yields a line segment \(LB\) that satisfies

\[(LB)^3 + DB \times (LB)^2 + (HB)^2 \times LB = (HB)^2 \times BG.\]

Figure 9: Khayyam’s construction, in which line segment \(LB\) is the solution to the cubic.

Khayyam includes with the construction geometric reasoning to verify this claim. Below is a version of the proof that follows closely a translation of his presentation (Fauvel and Gray, pp. 233-234.). This version outlines five key steps in the argument, with each step fully explained in turn. Below this geometric demonstration is then a suggestion for the reader to rework the argument in terms of contemporary algebraic notation perhaps better to understand the relationship between the line segment \(LB\) and the polynomial \(x^3 + ax^2 + b^2x = c^3\).

(1) First show that \(\triangle GLTK = \triangle ZABL\).
Figure 10: Since $\square ZAHT = \square GBHK$ and they share $\square LBHT$, then the areas of the green and purple rectangles are equal.

Apollonius established a property of hyperbolas stating that the product of the distances from any point on the curve to each of the asymptotes is a constant. Since $G$ and $Z$ are both on the same hyperbola with asymptotes $HK$ and $HB$, this means that the rectangles $\square ZAHT$ and $\square GBHK$ have the same area. Since these rectangles share the common rectangle $\square LBHT$, this means the difference rectangles $\square GLTK$ and $\square ZABL$ also have equal area.

(2) Next, develop the ratio $\frac{(LZ)^2}{(GL)^2} = \frac{(HB)^2}{(LB)^2}$.

Figure 11: A computed example of equal ratios in the case of $x^3 + 7.9x^2 + 1.3^2x = 2.1^3$. 
Since we have equal areas $\square GLTK = \square ZABL$, this means that $(LZ) \times (LB) = (GL) \times (LT)$. The sides of these rectangles are thus in proportion and furthermore $\frac{LZ}{GL} = \frac{HB}{LB}$ (since the length of $HB$ is equal to the length of $LT$). Likewise, then, their squares will also be in proportion, which gives the ratio for (2).

(3) Then, substitute to obtain $\frac{(LZ)^2}{(GL)^2} = \frac{LD}{GL}$.

Figure 12: The green and purple triangles are similar. This yields the ratio $\frac{LG}{LZ} = \frac{LZ}{LD}$, which is necessary for step (3).

Because $\triangle DGZ$ is inscribed on the diameter in a semi-circle, it is a right triangle by Thales Theorem. Then $\triangle LGZ$ and $\triangle LDZ$ are also right triangles and, in fact, these three triangles are similar. Thus, we have $\frac{LG}{LZ} = \frac{LZ}{LD}$ which gives the necessary piece for our substitution: $(LZ)^2 = GL \times LD$. The equality (3) follows directly.

(4) Next, demonstrate that $(HB)^2 \times GL = LD \times (LB)^2$.

Combine (2) and (3) above for $\frac{(LZ)^2}{(GL)^2} = \frac{(HB)^2}{(LB)^2} = \frac{LD}{GL}$. This means $(HB)^2 \times GL = LD \times (LB)^2$. Khayyam interpreted this as a geometric fact in terms of geometric solids, saying that the solid whose base is a square with sides of length of $HB$ and height $GL$ will have volume equal to a solid whose base is a square with sides of length $LB$ and height $LD$. 
Finally verify that the constructed segment $LB$ satisfies the cubic.

Since point $B$ is between $D$ and $L$, then $LB + DB = LD$. This means we can divide the box with base $(LB)^2$ and height $LD$ into two smaller boxes: the cube of $LB$ plus a box with base $(LB)^2$ and height $DB$. This, combined with (4) above, gives\[(HB)^2 \times GL = LD \times (LB)^2 = (LB + DB) \times (LB)^2 = (LB)^3 + DB \times (LB)^2.\]

Now add to each side the volume $(HB)^2 \times LB$ to get\[(HB)^2 \times GL + (HB)^2 \times LB = (LB)^3 + (LB)^2 \times DB + (HB)^2 \times LB.\]

Since $GL + LB = GB$, then\[(HB)^2 \times GB = (LB)^3 + DB \times (LB)^2 + (HB)^2 \times LB,\]
and the constructed segment $LB$ solves the cubic.

![Figure 13: Khayyam’s construction, in which line segment $LB$ is the solution to the cubic.](image)

**An algebraic rendering of the demonstration**

A geometric solution to a cubic equation may seem peculiar to modern eyes, but the study of cubic equations (and indeed much of medieval algebra) was motivated by geometric problems. Khayyam was nevertheless explicitly aware that the arithmetic problem of the cubic remained to be solved. He never produced such a solution; nor did anyone else until Gerolamo Cardano in the mid-16th century. Readers of this article may be particularly interested to see the relationships between a modern polynomial and Khayyam’s geometric treatment of a cubic.

Consider the polynomial $x^3 + ax^2 + b^2x = c^3$. To follow Khayyam’s demonstration, take $HB = b$, $DB = a$, and $BG = \frac{c^3}{b^2} = p$. Recall the hyperbolic property from Apollonius that states the product of the distances from any point on the curve to each of the asymptotes is a constant. If we take $(x, y)$ to be points on the hyperbola, then this property for points $G$ and $Z$ could be written as $pb = x(y + b)$. Readers may wish to work through the remainder of
Khayyam’s justification from this modern perspective.

**Shouldn’t a cubic have three roots?**

Notably, Khayyam’s construction generates a single line segment \( LB \) which solves the cubic of the form \( x^3 + C_1 x^2 + C_2 x = C_3 \). In the movie, for an example we have fixed the coefficients to be \( x^3 + 7.9 x^2 + 1.3^2 x = 2.1^3 \). The slider bars on the GeoGebra applet enable students to explore how this construction works for cubics with a range of coefficients. This dynamic exploration facilitates an appreciation of the general nature of the proof — Khayyam’s method is not simply constructing one root to one specific polynomial. His construction technique constructs one root for any polynomial of the form \( x^3 + C_1 x^2 + C_2 x = C_3 \), where \( C_1, C_2, C_3 > 0 \). The applet also presents in an accessible visual form the relationship between Khayyam’s segment \( LB \) and a root of the cubic graph familiar to many students. In a second applet, we have reflected Khayyam’s construction about \( HB \), the vertical asymptote of the hyperbola. Then, superimposing a Cartesian graph of a cubic shows clearing that \( BL \) corresponds exactly to the positive root of the cubic. But what of the other roots?

![Figure 14](image-url)

**Figure 14**: A reflection of Khayyam’s construction for \( x^3 + 4.3x^2 + 1.4^2 x = 1.5^3 \), superimposed with a Cartesian graph of the equivalent cubic function. We take \( HB \), an asymptote for the hyperbola, as a vertical axis and line segment \( BL \) has length \( x \), which is exactly the value of a root to the cubic polynomial.

The Medieval Islamic geometrical perspective ruled out negative coefficients and solutions, but this constructive technique from Khayyam is nonetheless capable of finding all real solutions.
to a cubic of the form \( x^3 + C_1 x^2 + C_2 x = C_3 \). A few additions to the construction generate the other two real roots (if they exist). First, we need to construct the complete circle on diameter GD and not simply the top semi-circle. Second, we need the other half of the hyperbola through point G with asymptotes HK and HB. The lower half of the hyperbola either intersects the circle at two points, at one point of tangency, or not at all. If the lower half of the hyperbola intersects the circle at two distinct points, the \( x \)-values corresponding to those intersection points are two negative roots to the cubic. If the lower half of the hyperbola intersects the circle at a point of tangency, the \( x \)-value of that point corresponds to a repeated negative root. If the lower half of the hyperbola does not intersect the circle, then there is a pair of complex roots. We know, by Descartes’ Rule of Signs, that polynomials of the form \( x^3 + ax^2 + b^2x = c^3 \) cannot have more than one positive root. Khayyam’s construction as presented finds a single, positive root and, if enhanced by completing the circle and hyperbola, gives complete information about all roots to cubics of the specified form.

Figure 15: A reflection of Khayyam’s construction for the cubic \( x^3 + 5.3x^2 + 1.85^2x = 1.84^3 \), enhanced with the complete hyperbola and both halves of the circle. The Cartesian graph of the cubic function \( f(x) = x^3 + 5.3x^2 + 1.85^2x - 1.84^3 \) is superimposed on the construction. This illustrates that the \( x \)-coordinates of the points of intersection (not along the \( x \)-axis) between the circle and the hyperbola are, in fact, roots to the cubic.

Khayyam set out to deal systematically with all fourteen unsolved types of cubic equations specified above. In other writings, he solves each one in sequence using similar constructions and justification based on intersecting conic sections. Demonstrations for several other of these constructive solutions are available online. Like the applet here, these all provide examples of how technology can facilitate student engagement with substantial questions about the history
of mathematics. The use of GeoGebra software reduces the difficulty of interpreting Khayyam’s static diagram, and dissecting the dependencies inherent in the construction to see how the technique can apply to every cubic of this form. Reflecting Khayyam’s construction and superimposing a Cartesian graph of the cubic also provides a useful connection between his unfamiliar approach to solving a cubic and the more familiar process of locating a root on a graph.

In a mathematical context where powers of $x$ literally corresponded to geometrical dimensions, solving cubic equations represented the height of mathematical achievement. But what did Khayyam achieve? Given the above observation, is it accurate to say that Khayyam did in fact, solve the complete cubic problem? Or did Cardano do something unimagined by Khayyam? In what ways are or are not these techniques addressing similar questions? Questions like these invite students to delve into issues of mathematical and historical context. What does it mean to attain a mathematical result? Investigating solutions to cubic equations that do not persist as the first general solution gives rise to explorations of innovation and novelty. This GeoGebra representation may provide students with an understanding of Khayyam’s marvelous construction that is adequate to engage these and other questions facilitating fruitful discussion in a history of mathematics classroom.
References


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